



## Slip line solution by spreadsheet

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### ABSTRACT

Many soil stability problems in geotechnical engineering can be analyzed by means of the classical slip line method – a relatively rigorous (baseline) solution. A slip line solution satisfies the lower bound theorem in plasticity theory, i.e. it satisfies boundary conditions, equilibrium equations and yield criteria for soil. It also provides insight into the soil failure mechanism, while requiring fewer assumptions than more traditional engineering approaches. Two decades ago computer programs were generally needed in order to obtain a solution of soil capacity by the slip line method, making the process difficult and tedious for others to follow. This paper proposes a practical method using spreadsheets for calculating soil capacity. Three typical example situations involving slope, tunnel, and foundation problems are solved using the slip line method and compared with established mathematical approaches. The procedure is based on using an iteration feature embedded in widely available spreadsheets, to process a finite difference matrix. Iterative searching of various solutions is performed automatically by invoking a spreadsheet's optimization tool, Solver. This spreadsheet-based method makes it easier to visualize multilayer iterations, and VBA user-functions reduce the programming effort needed.

### RÉSUMÉ

Beaucoup de problèmes de stabilité géotechnique peuvent être analysés au moyen de la méthode de lignes de glissement - une solution relativement rigoureuse. La solution lignes de glissement correspond au théorème de limite inférieure dans la théorie de plasticité, c.-à-d. elle satisfait les conditions extérieures, les équations d'équilibre et les critères de rupture du sol. Un autre avantage vient de l'évaluation du mécanisme d'échec de sol avec des prétentions moins intuitives impliquées que des approches approximatives de technologie plus traditionnelle. Les programmes machine il y a des décennies de couples étaient généralement nécessaires pour obtenir une solution de la capacité de sol par la ligne méthode de glissement, et il était pénible de suivre de d'autres. Dans le papier on propose une méthode pratique à l'aide des bilans pour calculer la capacité de sol. Trois problèmes typiques d'exemple comprenant la pente, le tunnel, et la base sont résolus respectivement et comparés à l'approche mathématique établie. Le procédé proposé est basé sur un dispositif de l'itération inclus dans des bilans largement disponibles dans l'application de l'armature finie de différence. La recherche itérative de diverses situations intéressées de solution est effectuée automatiquement en appelant le solutionneur d'outil d'optimisation d'un bilan. L'attraction de la méthode de bilan peut particulièrement être sur le transparent des itérations multicouche plus des utilisateur-fonctions de VBA trouvées sur le processus de solution numérique compliqué, qui est la plupart du temps une matité principale pour la programmation.

## 1 INTRODUCTION

This paper is concerned with the numerical solution of plane strain problems by the slip line method using a single page of spreadsheet. For 3D problems, the procedure is similar, except that multiple sheets are used, and the worksheets are linked to represent the behaviour of the individual slices or layers.

The slip line method is also known as the method of stress characteristics. It is aimed at solving the classical geotechnical bearing capacity problem of a rigid foundation resting on a cohesive-frictional soil mass and loaded to failure by a central vertical force. The soil mass is assumed as an ideally plastic material which behaves elastically until it yields and then undergoes deformation indefinitely at constant stress unless strain-strengthening or work-weakening occurs. This plastic behaviour is defined by a full set of constitutive equations. Although real soils will not correspond exactly to this model, it does provide a theoretical basis on which the actual behaviour of soil masses may be compared.

The theory is based entirely on plastic equilibrium from which the stress in the soil mass at failure is deduced. The failure regions do not need to cover the entire soil mass: however, the yield condition must be satisfied everywhere within the failure regions. Generally, routine highway embankment stability, strip foundation bearing capacity, retaining walls, tunnels and some slope failures all may be analyzed using the slip line method. Some analytical closed form solutions can be obtained by assuming the soil is weightless (Sokolovskii, 1965). Inclusion of soil weight considerably complicates mathematical solution so that numerical methods must be employed for most cases (Cheng et al. 2005). Numerical methods traditionally require computer programs, which, though useful (Martin, 2003), may obscure the underlying elegant concepts. Coding or mastering the details of such a program is likely to be too time-consuming for most practising geotechnical engineers.

A complete description of the slip line approach is found in Hill (1950), Wu (1966), Booker et al. (1977), Chen (1975), and Davis et al. (2002).

## 2 FINITE DIFFERENCE SOLUTION

Referring to the coordinate system shown in Figure 1, two equations of equilibrium under conditions of plane strain are established:

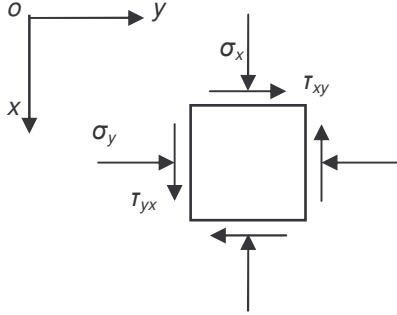


Figure 1. Stresses on an element in plane strain problem

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \gamma \quad [1]$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad [2]$$

where  $\gamma$  is the unit weight of the soil material in the  $x$  direction. The material is in a state of failure and the stresses obey the Mohr-Coulomb yield condition:

$$\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_x + \sigma_y}{2} \sin \varphi + c \cdot \cos \varphi\right)^2 \quad [3]$$

where  $c$  and  $\varphi$  are the cohesive strength and friction angle respectively. The stresses can be also expressed as follows:

$$\sigma_x = P + R \cos 2\theta \quad [4]$$

$$\sigma_y = P - R \cos 2\theta \quad [5]$$

$$\tau_{xy} = R \sin 2\theta \quad [6]$$

where angle  $\theta$  is the direction of the major principal stress  $\sigma_1$  relative to the  $x$  axis;  $P$  and  $R$  are the mean stress and the radius of Mohr stress circle, namely

$$P = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_3}{2} \quad [7]$$

$$R = P \sin \varphi + c \cos \varphi \quad [8]$$

By substituting Eqs [4] to [6] into equilibrium Eqs [1] to [2], we get the following governing equations:

$$\frac{\partial P}{\partial x} (1 + \sin \varphi \cos 2\theta) + \frac{\partial P}{\partial y} \sin \varphi \sin 2\theta + 2R \quad [9]$$

$$\left( -\frac{\partial \theta}{\partial x} \sin 2\theta + \frac{\partial \theta}{\partial y} \cos 2\theta \right) = \gamma$$

$$\frac{\partial P}{\partial y} (1 - \sin \varphi \cos 2\theta) + \frac{\partial P}{\partial x} \sin \varphi \sin 2\theta + 2R \quad [10]$$

$$\left( \frac{\partial \theta}{\partial x} \cos 2\theta + \frac{\partial \theta}{\partial y} \sin 2\theta \right) = 0$$

These are two simultaneous first order partial differential equations governing the variation of stress state in terms of  $(P, \theta)$ . Since they are hyperbolic, a suitable approach to solving them is the so-called method of characteristics (Hill, 1950; Abbott, 1966). In this paper, we assume the  $\alpha$ -line is the line which makes a clockwise acute angle with the direction of  $\sigma_1$  and the  $\beta$ -line which makes a counter-clockwise acute angle with the direction of  $\sigma_1$ . So the two characteristics ( $\alpha, \beta$ ) are defined by

$$\alpha\text{-line: } \frac{dy}{dx} = \tan(\theta - \mu) \quad [11]$$

$$\beta\text{-line: } \frac{dy}{dx} = \tan(\theta + \mu) \quad [12]$$

where  $\mu = \pi/4 - \varphi/2$ . Physically, the two characteristics represent the two failure planes on which the failure criterion is satisfied. Considering the directional derivative along the  $\alpha$ -line and  $\beta$ -line, respectively

$$\frac{\partial}{\partial S_\alpha} = \cos(\theta - \mu) \frac{\partial}{\partial x} + \sin(\theta - \mu) \frac{\partial}{\partial y} \quad [13]$$

$$\frac{\partial}{\partial S_\beta} = \cos(\theta + \mu) \frac{\partial}{\partial x} + \sin(\theta + \mu) \frac{\partial}{\partial y} \quad [14]$$

Therefore,

$$\sin 2\mu \frac{\partial}{\partial x} = \sin(\theta + \mu) \frac{\partial}{\partial S_\alpha} - \sin(\theta - \mu) \frac{\partial}{\partial S_\beta} \quad [15]$$

$$\sin 2\mu \frac{\partial}{\partial y} = -\cos(\theta + \mu) \frac{\partial}{\partial S_\alpha} + \cos(\theta - \mu) \frac{\partial}{\partial S_\beta} \quad [16]$$

By substituting Eqs [15] to [16] into equilibrium Eqs [9] to [10], we get the following governing equations:

$$-\frac{\partial P}{\partial S_\alpha} \sin 2\mu + 2R \frac{\partial \theta}{\partial S_\alpha} + \gamma(\sin 2\mu \frac{\partial x}{\partial S_\alpha} + \cos 2\mu \frac{\partial y}{\partial S_\alpha}) = 0$$

α - line [17]

$$\frac{\partial P}{\partial S_\beta} \sin 2\mu + 2R \frac{\partial \theta}{\partial S_\beta} + \gamma(-\sin 2\mu \frac{\partial x}{\partial S_\beta} + \cos 2\mu \frac{\partial y}{\partial S_\beta}) = 0$$

β - line [18]

The above two equations are the classical slip line equations, and can be solved using the finite difference method as suggested by Sokolovskii (1965). As shown in Figure 2, starting from two points *A* and *B* with known solutions the results for the intersection point *P* can be given. If the grid size is small enough, Eqs [17] to [18] can be rewritten as,

$$-\sin 2\mu(P_P - P_A) + (R_P + R_A)(\theta_P - \theta_A) = -\gamma \sin 2\mu(x_P - x_A) - \gamma \cos 2\mu(y_P - y_A) \quad [19]$$

$$\sin 2\mu(P_P - P_B) + (R_P + R_B)(\theta_P - \theta_B) = \gamma \sin 2\mu(x_P - x_B) - \gamma \cos 2\mu(y_P - y_B) \quad [20]$$

From Eqs [11] to [12], the following relations can be established:

$$y_P - y_A = \tan\left(\frac{\theta_P + \theta_A}{2} - \mu\right)(x_P - x_A) = \left(\frac{dy}{dx}\right)_1(x_P - x_A) \quad [21]$$

$$y_P - y_B = \tan\left(\frac{\theta_P + \theta_B}{2} + \mu\right)(x_P - x_B) = \left(\frac{dy}{dx}\right)_2(x_P - x_B) \quad [22]$$

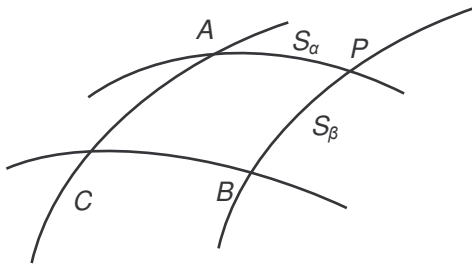


Figure 2. α and β lines in slip line solution

Based on the above four equations, we can get results for the point *P* as follows:

$$\theta_P = \frac{\gamma \sin 2\mu(x_A - x_B) - \gamma \cos 2\mu(2y_P - y_A - y_B) + \sin 2\mu(P_B - P_A) + (R_P + R_B)\theta_B}{R_A + R_B + 2R_P} \quad [23]$$

$$P_P = \frac{[\sin 2\mu(x_A - x_B) - (\theta_P - \theta_B) \sin \varphi]P_B - 2c + (\theta_P - \theta_B) \cos \varphi + \gamma \sin 2\mu(\theta_P - \theta_B) - \gamma \cos 2\mu(y_P - y_B)}{\sin 2\mu + (\theta_P - \theta_B) \sin \varphi} \quad [25]$$

$$x_P = \frac{y_B - y_A + \left(\frac{dy}{dx}\right)_1 x_A - \left(\frac{dy}{dx}\right)_2 x_B}{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2} \quad [26]$$

$$y_P = y_B + \left(\frac{dy}{dx}\right)_2(x_P - x_B) \quad [27]$$

Based on the boundary conditions, an iterative process is needed to obtain the values of  $x_P$ ,  $y_P$ ,  $\theta_P$  and  $P_P$ . And this iterative process is easily performed on a spreadsheet platform.

### 3 IMPLEMENTATION BY SPREADSHEET

Prior to starting to set up a spreadsheet for slip-line calculation, some preparation is required. On the **Tools** menu, click **Options**; on the **Calculation** tab, select **Manual Calculation** and **Recalculate before save** and make sure that **Iteration** is turned off.

The purpose of doing this first is to avoid a numerical problem (e.g division by zero) which would otherwise occur in the course of setting up the spreadsheet. Low et al. (1997) points out that "...iterative calculations among spreadsheet cells connected by circular references will produce a solution only if the iterative calculations lead to convergence." These adjustments can be reversed once the spreadsheet is complete. If the spreadsheet hangs, as shown by the appearance for example of "#" it needs to be debugged. For spreadsheets using an iterative finite difference solution for the slip line method, initially the calculation of Range **x** depends on both Range **y** and Range **θ**; Range **y** depends on both Range **x** and Range **θ**; Range **P** depends on all three Ranges **x**, **y** and **θ**. Therefore, Range **θ** is a critical path. When the spreadsheet hangs, turn off iteration and automatic recalculation, then try deleting Range **θ** and re-input its formulae correctly. Then turn iteration and automatic recalculation on again. It is helpful to practise first with a simpler sheet involving multilayer iteration. Reading Low's paper also helps.

Wherever optimization is required, do not hesitate to make use of the embedded Goal Seek or Solver.

### 3.1 Slope Stability for Landmark Project and Embankment

The main task in this case involves slope stability analysis to establish development setback lines and geotechnical design of a collector road along an escarpment (Hopkins et al., 2005).

When many buildings and retaining structures are constructed adjacent to a slope, the design of a strip-raft foundation close to the edge of the slope is a critical problem. The slope will fail when the vertical load acting on the slope crest reaches a critical value. Figure 3 shows a spreadsheet setup for a soil slope with an angle of friction  $\phi = 30^\circ$ ; cohesion  $c = 50\text{kPa}$ ; and unit weight  $\gamma = 0\text{ kN/m}^3$ . Calculation of the critical loads is implemented as follows:

- i. Input the soil physical parameters ( $c$ ,  $\phi$ ,  $\gamma$ ,...) and formula in cells A11 to H11, where the framed cell contains the formula;  $m_j$  is the number of  $\alpha$  lines;  $\epsilon$  is the direction of the slope surface relative to the  $y$ -axis, regarded as positive when pointing downwards to the right.; the "step" shown in cell H11 is the mesh step along the slope surface; "step1" in cell G11 is the distance between the first  $\alpha$ -line and singularity "0.0", which is the origin of the coordinate system. The step is chosen so as to obtain a suitable degree of accuracy.
- ii. Create four user-defined spreadsheet functions shown as follows, by clicking Insert/Macro/module in Microsoft Excel and then, on the module sheet, type "Function xP.....", etc. The four functions xP(), yP(), pP(), ThetaP() are based on Eqs [23] to [27] and shown below; then turn iteration on.
- iii. Enter constants and formulae into the columns labelled  $x$  as follows:  
 $E17=IF(H12=0,0,F16+I12*SIN(RADIANS(H12)));D18=IF(H12=0,0,E17+J12*SIN(RADIANS(H12)));C19=IF(H12=0,0,D18+J12*SIN(RADIANS(H12)));B20=IF(H12=0,0,C19+J12*SIN(RADIANS(H12)));$  other boxed cells input 0; all non-boxed cells in  $x$  contain formulae xP(), for example  $F17=xP(\mu,F23,E24,F16,E17,F38,F37,E38)$ , then autofill all non-boxed cells left in  $x$ .
- iv. Enter constants and formulae into the columns labelled  $y$  as follows:  
 $E24=IF(H12=0,F23+I12,F23+I12*COS(RADIANS(H12)));D25=IF(H12=0,E24+J12,E24+J12*COS(RADIANS(H12)));C26=IF(H12=0,D25+J12,D25+J12*COS(RADIANS(H12)));B27=IF(H12=0,C26+J12,C26+J12*COS(RADIANS(H12)));$  other boxed cells are input 0; all non-boxed cells in  $y$  contain formulae yP(), for example  $F24=yP(\mu,E24,E17,F17,F38,E38)$ , then autofill all the non-boxed cells left in  $y$ .
- v. Enter constants and formulae into the columns labelled  $P$  as follows: all boxed cells contain formulae as shown in Figure 3; all non-boxed cells in  $P$  contain formulae pP(), for example  $F31=pP(\gamma,\mu,\phi,c,E38,F38,E31,E17,F17,E24,F24)$ , then autofill all the non-boxed cells left in  $P$ .

- vi. Enter the data and formulae into the columns labelled as follows: all boxed cells contain formulae as shown in Figure 3;

$$G37=F37-(F37-I37)/3$$

$$H37=G37-(F37-I37)/3$$

$$F38=E38+(\$J\$38-\$E\$38)/(mj+1)$$

$$G38=F38+(\$J\$38-\$E\$38)/(mj+1)$$

$$H38=G38+(\$J\$38-\$E\$38)/(mj+1)$$

$$I38=H38+(\$J\$38-\$E\$38)/(mj+1)$$

Other non-boxed cells contain formulae ThetaP(), for example

$$E39=ThetaP(\gamma,\mu,\phi,c,E17,D18,E24,D25,E25,E31,D32,E32,E38,D39)$$
, then autofill all non-boxed cells left in  $\theta$ .

The VBA functions are as follows:

Function xP(mu, yA, yB, xA, xB, ThetaP, ThetaA, ThetaB)

$$dy1 = \tan(0.5 * (\text{ThetaP} + \text{ThetaA}) - \mu)$$

$$dy2 = \tan(0.5 * (\text{ThetaP} + \text{ThetaB}) + \mu)$$

$$t1 = yB - yA + dy1 * xA - dy2 * xB$$

$$t2 = dy1 - dy2$$

$$xP = t1 / t2$$

End Function

Function yP(mu, yB, xB, xP, ThetaP, ThetaB)

$$dy2 = \tan(0.5 * (\text{ThetaP} + \text{ThetaB}) + \mu)$$

$$yP = yB + dy2 * (xP - xB)$$

End Function

Function pP(gamma, mu, phi, c, ThetaB, ThetaP, pB, xB, xP, yB, yP)

$$\mu = 2 * \mu$$

$$\phi = \phi / 180 * 3.1415926$$

$$t1 = (\sin(\mu) - (\text{ThetaP} - \text{ThetaB}) * \sin(\phi)) * pB$$

$$t2 = -2 * c * (\text{ThetaP} - \text{ThetaB}) * \cos(\phi)$$

$$t3 = \gamma * \sin(\mu) * (xP - xB)$$

$$t4 = -\gamma * \cos(\mu) * (yP - yB)$$

$$t5 = \sin(\mu) + (\text{ThetaP} - \text{ThetaB}) * \sin(\phi)$$

$$pP = (t1 + t2 + t3 + t4) / t5$$

End Function

Function ThetaP(gamma, mu, phi, c, xA, xB, yA, yB, yP, pA, pB, pP, ThetaA, ThetaB)

$$\mu = 2 * \mu$$

$$\phi = \phi / 180 * 3.1415926$$

$$rP = pP * \sin(\phi) + c * \cos(\phi)$$

$$ra = pA * \sin(\phi) + c * \cos(\phi)$$

$$rb = pB * \sin(\phi) + c * \cos(\phi)$$

$$t1 = \gamma * \sin(\mu) * (xA - xB)$$

$$t2 = -\gamma * \cos(\mu) * (2 * yP - yA - yB)$$

$$t3 = \sin(\mu) * (pB - pA)$$

$$t4 = (rP + ra) * \text{ThetaA}$$

$$t5 = (rP + rb) * \text{ThetaB}$$

$$t6 = ra + rb + 2 * rP$$

$$\text{ThetaP} = (t1 + t2 + t3 + t4 + t5) / t6$$

End Function

Sokolovskii (1965) had obtained analytical closed form solutions (Figure 3) for calculation of the critical loads for weightless soil slope cases. For this example, the analytical solution shown in Figure 3 is 334.81 kPa and the numerical solution using the spreadsheet is 334.97 kPa.

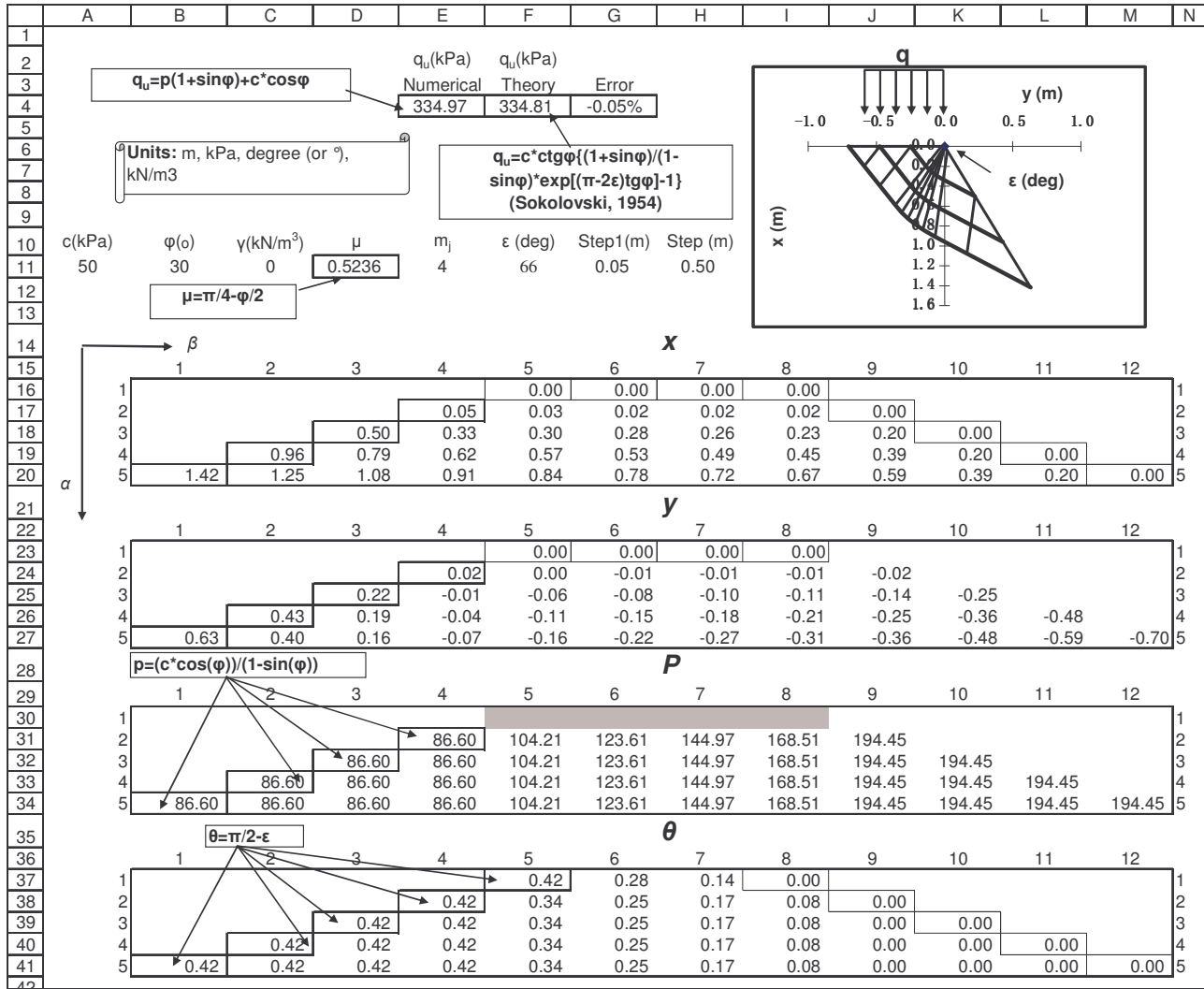


Figure 3. Spreadsheet solution, compared with the analytical solution by Sokolovskii (1965)

### 3.2 Bearing Capacity of Footing

Bearing capacity is a classical problem that has been considered by Hill (1950), Sokolovskii (1965), and Chen (1975) using plasticity theory, the slip line method and limit analysis. If  $\epsilon$  in the first example is equal to 0, the spreadsheet setup for a soil slope in Figure 3 can then be used to analyze the bearing capacity of a footing. For improving the accuracy, a small grid size is used (Figure 4). The soil is homogenous with cohesion  $c = 144.5$  kPa, and unit weight  $\gamma = 17.01$  kN/m<sup>3</sup>. The width of the footing is 17m. By changing the cell "step" until the longest  $\alpha$ -line runs through the midpoint of the base, the ultimate bearing capacity can be obtained. Some formulations have been proposed to determine the bearing capacity, including Terzaghi (1943), Meyerhof (1951), Vesic (1973) and Chen (1975), among others. The numerical results in this paper are compared with the classic formulations above and different friction angles are considered. Because the slip line solution satisfies

the equations of equilibrium, the results are among the best lower bound results available. This is illustrated in Table 1.

### 3.3 Circular Tunnel – Thickness of Plastic Zone

A plane strain problem of a deep circular tunnel is considered here. If the *in situ* stress is sufficiently great there will be a failure region surrounding the tunnel. The tunnel excavation surface and the outer surface of the failure region surrounding the excavation are both principal surfaces. In fact, all circular surfaces concentric with the excavation surface are principal surfaces provided they are within the plastic zone. If we know the tunnel radius, the internal support pressure and the pressure acting on the boundary of the failure region, then the radius of the plastic zone around the tunnel may be obtained (Davis et al., 2002).

In this example, the  $\alpha$ -lines are concave towards the left. The  $\delta$  is an angle which describes the location where



the outermost slipline daylight on the excavation surface, let us call it point A, as shown in Figure 5. In order to find the thickness of the plastic zone, we vary the  $\delta$  value until cell U20 equals to zero. Figure 6 shows a spreadsheet setup for this case. This can be implemented easily by using solver.

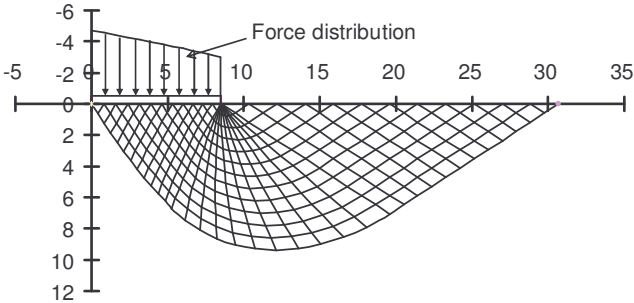


Figure 4. Spreadsheet results for bearing capacity of footing

Table 1. Comparison of the slip line solution with other classic formulations for bearing capacity

$\phi$ (°)	$x_{ls}$ (slipline)	Vesic (1973)	Meyerhof (1951)	Terzaghi (1943)	Chen (1975)
0	742.9	743.0	743.0	743.0	743.0
5	972.1	1002.6	947.7	950.0	1004.9
10	1303.6	1382.7	1256.8	1273.3	1395.5
15	1802.5	1968.7	1749.2	1791.1	2011.0
20	2587.7	2921.9	2558.4	2654.8	3039.0
25	3894.9	4565.7	3971.7	4166.0	4867.1
30	6214.8	7595.3	6619.2	6968.2	8352.7

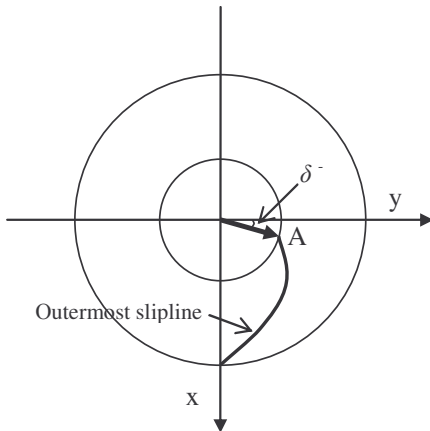


Figure 5. Definition of skew angle  $\delta$

4 SUMMARY

The slip line method is a baseline calculation tool which can be used to analyze many geotechnical failure problems. The method is regarded as difficult to apply in

routine geotechnical practice mainly because of the multilayer iteration involved. However, with the iterative features now available in spreadsheets complete iterative finite difference procedures can be carried out relatively easily. The examples in this paper give the same results as those based on closed-form solutions, and benefit from the transparency of operation of the spreadsheet-based approach. Spreadsheet setups could potentially handle more complex cases, e.g., loaded sloping ground with or without setback (Cheng et al., 2005), dynamic loads, pore water pressure, anisotropy, non-linearity, and probabilistic analysis in engineering practice. Some special problems may be solved in the same manner, for example, fall cone tests (Houlsby, 1982). What needs to be borne in mind is that the slip line solution is always a lower bound (although not guaranteed to be the lowest), so in general it provides a conservative estimate of capacity or stability. If we can also apply an upper bound method (e.g. Sarma, 1973) to the same problem, we can bracket the exact solution. Electronic copies of the spreadsheets are available from the authors on request.

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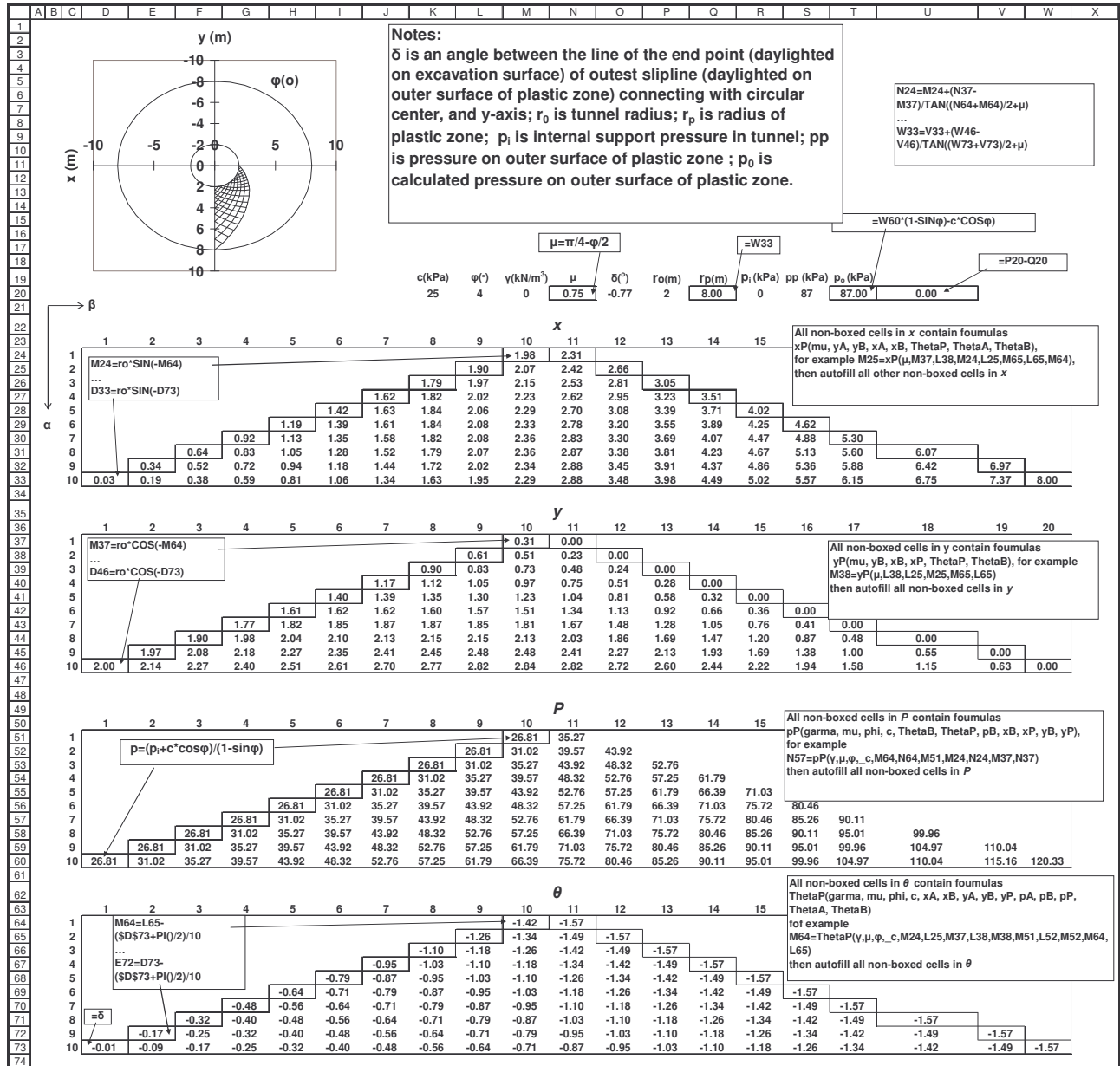


Figure 6. Tunnel example setup